

Intro: The 1-~~2~~-3 of modular forms

Congruences: some examples

For $n \geq 1$, $p(n) := \# \text{ partitions of } n$

Thm (Watson 1938): If $24n \equiv 1 \pmod{5^m}$ then

$$p(n) \equiv 0 \pmod{5^m}$$

For $n \geq 1$, $c(n) := \dim$ (degree $(n+1)$ piece of the Monster vertex algebra)

$$c(1) = 196884$$

Thm (Lehmer 1949): If $n \equiv 0 \pmod{2^m}$ then

$$c(n) \equiv 0 \pmod{2^{3m+8}}.$$

$$\left(\frac{1}{q}\right) + 744 + \sum_{n=1}^{\infty} c(n) q^n =: j(q)$$

j-invariant
(non-holom @ ∞)
modular form of weight 0

$$\sum_{n=1}^{\infty} p\left(\frac{n+1}{24}\right) q^n = \frac{1}{q^{1/24} \prod_{n=1}^{\infty} (1-q^n)}$$

$p(\text{non-integer}) = 0$

!!
Dedekind eta function,
modular form of weight $\frac{1}{2}$

There are congruences between coeffs of modular forms

$$\Delta := \eta^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

Δ modular form
of wt 12

$$\tau(n) \equiv n \sigma_9(n) \pmod{7}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

$$\sigma_9(n) = \sum_{d|n} d^9$$

e.g. if n is prime

$$\sigma_9(p) = 1 + p^9$$

Serre & Swinnerton-Dyer 1972

Modular form of weight k :

$$\mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$$

$f: \mathcal{H} \rightarrow \mathbb{C}$ analytic

$M_k \supset S_k$
cusp forms
 $z \in \mathcal{H}$

$$\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

$$f(z+1) = f(z)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

Fourier exp

and f is holomorphic @ ∞

$$f(z) = \sum a_n e^{2\pi i n z}$$

Eisenstein series:

$$E_k = \text{constant term} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

wt $k, k \geq 4$

$$\Delta = \frac{E_4^3 - E_6^2}{12^3}$$

weight 12

$$q = e^{2\pi i z}$$

Harder:

$$f = \frac{E_6 E_4^4 - E_6^3 E_4^3}{12^3} = q - 288q^2 - 12884q^3 + \dots$$

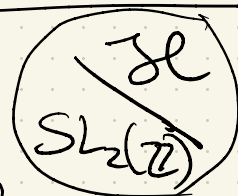
wt 22
 $f \in S_{22}$

$S_k \hookrightarrow T_p$ for each prime p Hecke operators

$f \in S_{2k}$ automatically an eigenvector
1-dim
eigenvalue of T_p is the p^{th} coefficient

modular forms "live on" the space

→ sections of line bundles



Riemann sphere without a point.

$$\mathcal{H}_2 = \left\{ \underbrace{Z \in M_2(\mathbb{C})}_{\substack{\uparrow \\ \text{3-dimensional} \\ \text{complex manifold}}} \mid \underbrace{Z^t = Z}_{\substack{\uparrow \\ \text{positive definite}}} , \underbrace{\operatorname{Im}(Z) > 0}_{\substack{\uparrow \\ \text{positive definite}}} \right\}$$

3-dimensional complex manifold.

$$f: \mathcal{H}_2 \rightarrow \mathbb{C} \text{ analytic}$$

Siegel modular forms
of genus 2

$$f\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} (z) \begin{pmatrix} C & D \end{pmatrix}^{-1}\right) = \det \begin{pmatrix} C & D \end{pmatrix}^k f(z)$$

More generally, take a representation
of $GL_2(\mathbb{C})$

$$f: \mathcal{H}_2 \rightarrow V \text{ analytic}$$

$$z \in \mathcal{H}_2 \quad (z, V)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{C})$$

$$\tau = \det^k$$

$$f\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} (z) \begin{pmatrix} C & D \end{pmatrix}^{-1}\right) = \tau \begin{pmatrix} C & D \end{pmatrix} f(z)$$

Conjecture (Harder) Given $f \in S_{22}$, ^{magic occurs} there exists a Siegel modular form F of weight $\tau = \text{Sym}^4 \otimes \det^{10}$ such that

$$\underline{\lambda(p)} \equiv \underline{a_p} + p^{13} + p^8 \pmod{41} \quad \text{for all primes } p$$

$\lambda(p)$ = Hecke eigenvalue for F

Bergström-Dummigan
Faber-vanderGeer

a_p = Hecke eigenvalue for f

GL_2

GSp_4

Chenevier-Lannes : proved Harder's conjecture